THE POSITIONING PROBLEM - A DRAFT OF AN INTERMEDIATE SUMMARY

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AN INFORMAL INTRODUCTION

The positioning problem arises when it is necessary to locate a set of geographically-distributed objects using measurements of the distances between some object pairs. In a Packet Radio Network, for instance, any two network members that can talk to each other may use a simple time-stamping mechanism to measure the distance between them; a distance measurement protocol may then be developed. The problem is whether and how the distance measurements can be used to determine the geographical location with respect to a given system of coordinates.

A knowledge of the precise location of each network node is crucial to the operation of Distributed Sensors Networks. The data collected and interpreted by different sensors may be correlated and integrated only if we know their precise location. A position-locating system may be invaluable to the operation of a fleet of vehicles, each equipped with a Packet Radio Unit. For example, monitoring the location of a fleet of security vehicles, aircraft, a tank division, or a flock of missiles could all be assisted by a position-locating system. Clearly a positioning system would be an important service to Packet Radio Network users.

A few problems must be solved before a good positioning system may be developed:

1. Efficient algorithms to determine the location of objects by using distance measurements should be developed.

2. Conditions under which a solution exists or does not exist should be identified.

3. Conditions under which there exists a unique solution should be established.

4. Conditions under which there exists a finite number of solutions should be identified. It should also be understood how to transform one solution into another.

5. Conditions under which the solution is insensitive to small measurement errors should be established.

6. Tight bounds upon the accuracy of the solution should be determined.

7. Ill-conditioned problems should be identified.

However, while the formulation of the problems is simple, the mathematical and algorithmic intricacies of deriving solutions are perplexing.

To develop some insight into the problems, let us consider a few simple examples. The simplest positioning problem of interest is to locate three points using distance measurements. By means of simple trigonometry, this may usually be done easily. However, let us consider a degenerate triangle (Figure 1). Because the system is very sensitive to errors, a small error in the measurement may produce a large error in the computed position. Some of the questions to be addressed are as follows: Why is the degenerate triangle sensitive to errors? How can we determine whether or not other systems are sensitive?

Figure 1. A Degenerate Triangular System

Positioning systems may be constructed by a simple procedure of pasting triangles together, and such systems may be positioned by solving the triangles from which they are constructed. For instance, consider the system of points depicted in Figure 2. It is possible to locate the points in the order numbered. However, the same system admits a few solutions (the number of which grows exponentially with the number of nodes). If we had some further information about the positions of the objects, how could we use it to identify the true solution? For instance, if one node is known to be a vehicle moving on a certain road, many of the feasible solutions that satisfy the distance constraints can be eliminated, because they assign the vehicle to a position not on the road. How should that elimination be effected?

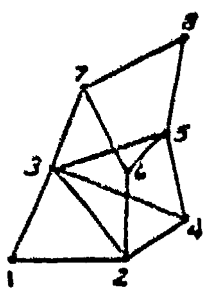

Figure 2. A Triangulated Positioning System

In particular, if a new measurement were given (Figure 3), how can we find the right solution among all possible ones? (A few solutions may match the measurements to within a given error.) We have shown that this last decision problem belongs to the class of difficult combinatorial decision problems--the so-called NP complete problems. The last statement also proves that the existence problem (i.e., Does a solution exist?) is NP complete.

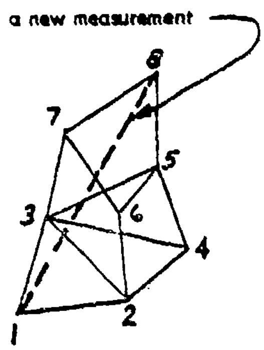

Figure 3. A Triangulated Positioning System with an additional Measurement

Not all positioning systems may be solved with the aid of triangles. In fact, for a system possessing a sufficiently large number of nodes, it is always possible that the position of the nodes can be located only by solving for the location of all points simultaneously. Such unfortunate systems require an enormous amount of computation. For a simple example, consider the hexagon of Figure 4. It is impossible to solve the location of its nodes using an incremental algorithm; all must be solved at once.

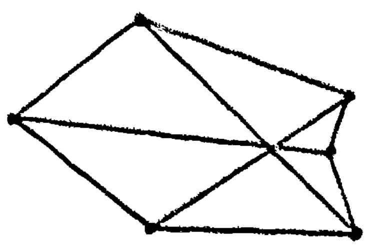

Figure 4. A Positioning System which may only be solved simultaneously

Fortunately enough, such primitive systems (whose parts cannot be positioned unless the whole system is) seem to be rare. Many positioning systems could be solved using an incremental process (which simplifies the solution algorithm and increases its speed and accuracy). However, an algorithm that would construct the location of a given point system by constructing subsystems first should be able to identify constructible parts. In particular, such an algorithm should be able to decide whether or not a given positioning system contains a subsystem that may be solved independently. Namely, any incremental construction algorithm must decide whether or not a given positioning system is primitive (i.e., constructible but having no constructible subsystems). The last problem seems to be a difficult combinatorial problem which we suspect to be NP complete (though we do not yet know how to prove this).

If the last conjecture is true, then the problem of constructing solutions to the positioning problem, using an incremental algorithm, is NP complete. This unfortunate result does not imply that "brute force" (i.e., iterative algorithms) should be preferred. It is reasonable to believe that most actual positioning problems may be better solved by means of an intelligent incremental algorithm. The precise meaning of "most" is yet to be defined.

Numerous other challenging and interesting related problems exist. While we will not make a comprehensive presentation, we will examine some of the problems formally, expose the difficulties, and present some partial solutions we have developed. This report is an extended summary of our present state of knowledge. A more detailed report is now being prepared.

1. THE PROBLEMS

The positioning problem can be described as follows:

1. $P \triangleq \{P_1, P_2, \ldots, P_N\}$, a set of points in the plane.

2. A set of distance measurements between some pairs of points. Each measurement datum consists of the identity of the pair P_i and P_j, the measured distance d_{ij} and an estimate of the measurement error ϵ_{ij}.

3. Position coordinates for at least three points, say P_1, P_2, P_3, to be called the <u>base triangle</u>.

We shall call the set of points P, together with the distance measurements $\{d_{ij}\}$ and the base triangle, a <u>point system</u>.

1.2 PROBLEMS

A <u>feasible position</u> of the point system is a set of coordinates that satisfies the distance constraints and assigns to the base triangle its actual coordinates. The set of all feasible positions will be called the <u>solution set</u> of the positioning problem. The positioning problem consists of characterizing the solution set namely,

1. Is the solution set empty? (<u>existence</u>)

2. Does the solution set contain a continuum of solutions, or is it a discrete set? Is it a finite set? (<u>generalized uniqueness</u>)

3. If the solution set is finite, what are all the feasible solutions? (position construction)

4. What is the error in the position due to propagation of measurement errors? (error analysis)

5. What is the sensitivity of the solution set to measurement errors? (sensitivity analysis)

In what follows we examine some partial answers to some of the difficult problems posed above.

Note that in the sequel we restrict ourselves to the problem of positioning points relative to each other, i.e., with respect to any coordinate system of our choice. This leaves us three degrees of freedom (two for translation and one for rotation) and the orientation for our choice of the coordinate system. With the aid of the third item on the input list (section 1.1) it is possible to position the point system absolutely once it has been positioned relative to an arbitrary coordinate system.

2. GEOMETRIC RESULTS

We associate with the point system a graph whose vertices represent points and whose edges represent distance measurements. We call this graph the measurements graph and say that a given property of a point system is combinatorial when it can be expressed in terms of the measurements graph only. A structure is a graph together with a mapping of edges into positive real numbers which we call lengths. A positioning system may be considered as a pin-jointed bar structure, i.e., a truss. Problems of uniqueness of (i.e., structure of the solution set) for the positioning problem translate into problems of rigidity of the respective truss. Problems of a solution's sensitivity to errors translate into problems of infinitesimal rigidity of the respective truss (i.e., admissibility of infinitesimal flexing of the truss). Problems of constructing a solution to the positioning problem correspond to construction of the truss. Therefore we shall use methods and terminology that pertain to both structures and trusses.

2.1 RIGIDITY

The results in this area fall into three classes:

1. A number of different characterizations of infinitesimal rigidity (error sensitivity).

2. An algorithm to determine whether or not a given solution of the positioning problem is sensitive to measurement errors.

3. A combinatorial characterization of plane rigidity in terms of a property of the underlying measurements graph.

While the problem of structural rigidity has attracted mathematicians, engineers, and architects for several centuries,[1] the solution to most fundamental questions of rigidity are far from known. Recently interest in this age-old problem has been renewed [WHITELY 77,78], and some significant contributions produced.

2.1.1 Characterization of rigidity

The first problem, i.e., that of characterizing rigidity, has a few solutions, all of which employ local infinitesimal characterizations. Loosely speaking, a structure is rigid if it does not admit relative motions of its parts, that is, if the only motions which it admits are trivial (i.e., translations and rotations). Therefore the study of rigidity is a study of possible motions. A rigid structure corresponds to a positioning problem with a discrete solution set. An infinitesimally rigid structure corresponds to an error-insensitive solution.

There are two approaches to motions of structures: It is possible to consider the velocity vectors of the nodes or the relative angular motion of edges attached to a common node. Accordingly, it is possible to develop two notions of infinitesimal rigidity. Another possible approach is to consider the stresses in the structure resulting from applying external forces. Rigidity may be defined as the ability of the structure to resolve forces. It is possible to show that both the above approaches are equivalent [GLUCK 75, WHITELY 77, 78].

Open problems:

1. Characterization of rigid structures which are not infinitesimally rigid.

   An infinitesimally rigid structure is rigid, but not vice versa. It is possible for a structure to be rigid (i.e., admit no finite relative motions of its parts) yet to admit infinitesimal perturbations (i.e., be sensitive to errors). A typical example of an error-rigid rigid structure is the degenerate triangle in Figure 1.

   The difficulty in solving this problem is that we do not possess extensive tools for global analysis, while local analysis is well developed.

2. Characterization of rigidity with respect to discontinuous motions such as reflections.

   Consider the pair of pasted triangles in Figure 5(a) below. The two triangles may be positioned with respect to each other in two distinct ways. The two resulting structures are rigid (do not admit non trivial motions) but admit relative reflection of parts. Other structures

[1] A partial list of researchers interested in the problem includes Pascal, Euler, Cauchy, Maxwell, Cayley, Alexandrov, and others.

admit a more complex form of discrete movement of their internal parts., e.g., Figure 5(b).

The problem of characterizing rigidity with respect to discrete motions is difficult, for we not only have to address a problem of global analysis, but also face diff'cult problems of combinatorial topology.

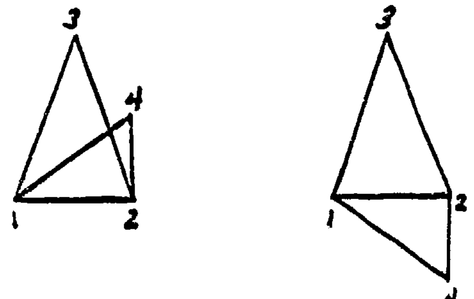

Figure 5(a)

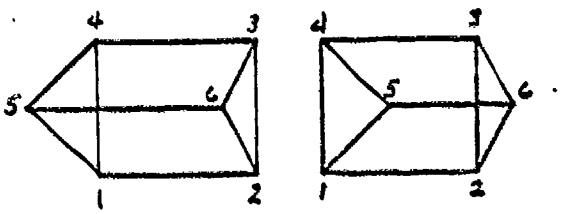

Figure 5(b)

Figures 5(a) and (b). Two cases of two rigid structures solving the same positioning problem.

2.1.2 Error sensitivity

The different definitions of infinitesimal rigidity induce different algorithms to determine whether or not a given structure is infinitesimally rigid. However, most of those algorithms have been produced by and for mathematicians unconcerned with computational efficiency. We have developed a novel algorithm to test whether a given feasible solution of the positioning problem is infinitesimally rigid, i.e., insensitive to errors.

The idea behind the rigidity testing algorithm is simple: try to solve for an admissible assignment of infinitesimal velocities that flexes the structure. It is necessary to examine only the effects of a velocity assignments over a set of basic circuits of the underlying measurement graph in order to reduce the problem to the solution of a linear system of equations.

<u>Open problems:</u>

1. Establish measures of error sensitivity and algorithms to compute sensitivity.

   The characterization of error sensitivity in terms of infinitesimal rigidity is too crude. We would like to have an estimate of <u>how much</u> error sensitivity a given structure possesses.

2. Develop sensitivity measures in terms of the distance measurements data, not the particular solution they yield.

   In addition to the difficulty involved in developing apriori sensitivity measures, we face the difficulty that the same measurement data may have a number of solutions, each possessing different sensitivity. Apriori it is even possible for a given system of measurement data to possess some rigid and some nonrigid solutions. Figure 6 depicts two solutions of the same positioning problem, one rigid and the other infinitesimally flexible.

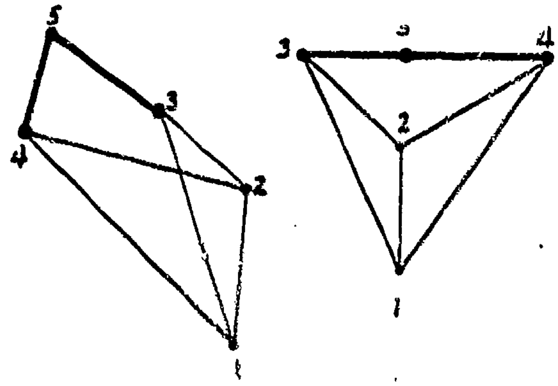

Figure 6. A positioning problem possessing a rigid and an infinitesimally-flexible solution

We shall now describe the third class of results in more detail.

2.2 FROM GEOMETRY TO COMBINATORICS

2.2.1 Stiff graphs

The most powerful results of the study of rigidity appear to be combinatorial characterizations of rigid structures. The ideas behind the passage from geometry to combinatorics are founded in some simple intuitive experiences,[2] the major idea of which is that some combinations of bars (edges) and hinges (nodes) are rigid for almost any choice of plane imbedding. For instance, the full graph on three nodes is rigid for all plane imbeddings; it is not infinitesimally rigid when the three points are colinear. Similarly, structures whose

[2] Intuition should not be pursued blindly, however, when it comes to problems of rigidity, for—as we shall see—many intuitive expectations turn out, surprisingly, to be false.

underlying graph is a full graph are rigid for almost any plane imbedding. On the other hand, some graphs will produce flexible structures for almost any plane imbedding (e.g., a circuit on four nodes is almost always flexible, except when one of the edges is assigned a zero length or when all four points are colinear). The problem is: <u>Is it possible to characterize graphs almost all of whose embeddings form rigid structures?</u>. This question has been the major problem of the theory of rigidity.

We shall define a critical combinatorial property of graphs which we call (plane) <u>stiffness</u>. First we associate with a graph $G \triangleq \langle V,E\rangle$ a number $f(G) \triangleq 2|V|-|E|-3$, measuring the overall excess of unknowns over constraints. (Each vertex contributes two unknown coordinates, and each edge constrains the two vertices incident unto it through a single equation for the distance. Note that we discount three degrees of freedom to account for possible external motion, i.e., translation and rotation.) The quantity $f(G)$ measures the overall <u>freedom of internal movement</u> of the graph.

The quantity $f(G)$ may be used to express the property of stiffness, i.e., of having a sufficient number of constraints to prevent relative motions of different parts of a graph. Loosely speaking, a graph is stiff if it is possible to remove some redundant edges so that the remaining graph has 0 degrees of internal freedom. and non of its subgraphs has an excess of constraints (i.e., a negative internal freedom). Formally, a graph $G=\langle V,E\rangle$ is <u>stiff</u> if it has a spanning subgraph $G'=\langle V,E'\rangle$ (i.e., G' is generated by removing excessive constraints from G) such that

1. $f(G')=0$ (i.e., G' has 0 degrees of internal freedom)

2. if G'' is any subgraph of G' then $f(G'') \geq 0$. (i.e., G' does not possess internally over-constrained subgraphs).

2.2.2 The rigidity theorem

The most important result of the geometric theory of positioning is

THEOREM (Plane Rigidity) [3]:

> <u>A rigid structure must have a stiff measurements graph. Conversely, almost all plane imbeddings of a stiff graph are rigid.</u>

(Here "almost all" is used in the topological sense, i.e., the set of rigid plane imbeddings of a stiff graph is open and dense in the space of imbeddings, which further implies that for any Borel probability measure on the space of all imbeddings, continuous with respect to Lebesgue measure, the set of non rigid imbeddings of a stiff graph is of measure 0.)

[3] [LAMAN 70, GLUCK 75, WHITELY 78, ROTH-ASIMOW 78]

The theorem above is a significant tool for handling positioning problems, namely, it makes it possible to infer properties of structure and derive answers to the problems of positioning by <u>examining the measurement graph only</u>. The results that we derive will be true for almost all assignments of distances to edges of the measurement graph, which greatly simplifies the study of the positioning problem.

To further appreciate the power of the result let us note in passing that the theorem does not generalize even to three dimensional structures. That is, it is possible to have a sufficient number of well distributed constraints and yet have a structure with a continuum of solutions no matter what lengths are assigned to the edges. Figure 7 below depicts a typical system. The combinatorial characterization of rigid structures in spaces of dimensions greater than two is an open problem.

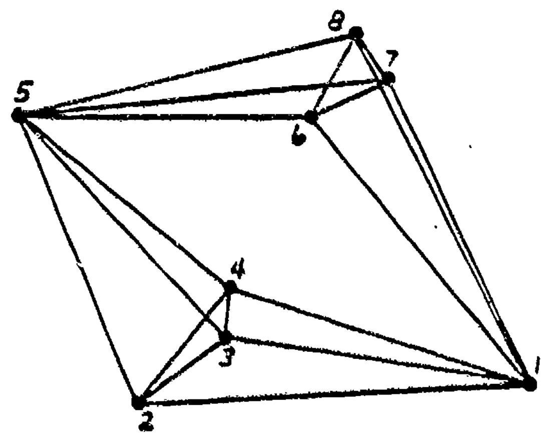

Figure 7. A counter example to a 3-dimensional rigidity theorem

2.3 LIMITATIONS OF THE RIGIDITY THEOREM

The rigidity theorem guarantees that a structure based on a stiff graph will almost always be rigid. However, it is possible to assign lengths to edges of a stiff graph such that the resulting structure admits infinitesimal flexing and is, therefore, very sensitive to errors. In fact, the structure becomes an error-increasing mechanism. One such structure is the degenerate triangle of Figure 1, another is Pascal's hexagon depicted in Figure 3. This hexagon is stiff and thus rigid for almost all plane imbeddings. However, the hexagon is infinitesimally flexible whenever (and only if) its nodes lie on a conic section. This bizarre result, due to [CROFTON 1878], has been rediscovered independently by many researchers (including ourselves). The proof follows from a simple application of a celebrated theorem of Pascal.

An even worse case is that of a stiff graph admitting a continuous motion (i.e., an imbedding of the stiff graph that is not even globally rigid). Such a structure is depicted in Figure 9.

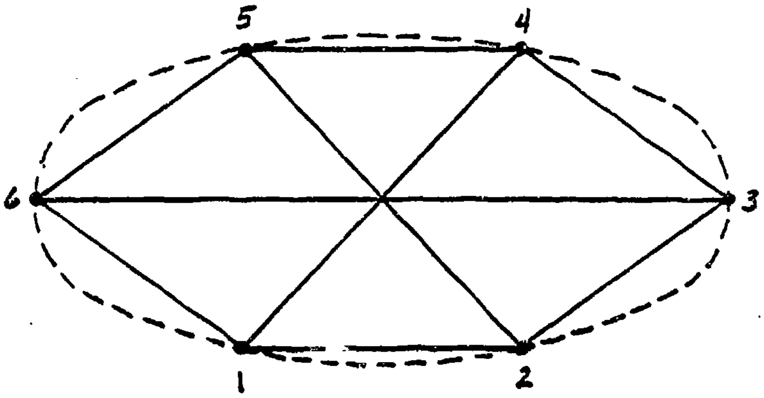

Figure 8. Pascal's Hexagon

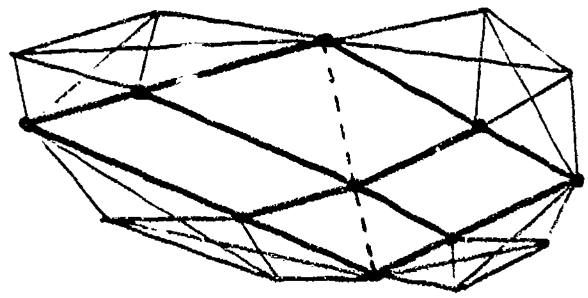

Figure 9. A flexible structure whose underlying graph is rigid.

Although the underlying graph is stiff (even overconstrained), the solution set of the respective positioning problem contains a continuum of solutions.

To produce such an example we started with a nonrigid structure (i.e., a mechanism), then meticulously added bars that did not constrain the motion of the original mechanism. Specifically, we started with a four-bar mechanism, selected a point on any diagonal and connected this point to points on the original mechanism. By careful selection of the connections to produce two parallelograms, the motion of the original mechanism is not perturbed by the additional "constraints." Additional bars are used only to hold each of the original four bars together.

The rigidity theorem guarantees that such unpleasant positioning systems are extremely rare. Yet we should bear in mind that they may exist (highly symmetric structures are very likely exceptions).

3. COMBINATORICS OF POSITIONING

Having seen the significance and limitations of stiffness, we will now study stiff graphs in order to develop methods for recognizing stiffness and for tearing structures into stiff subparts, later cementing those parts together. (Tearing is the natural instrument for a divide-and-conquer approach to the construction problem.)

3.1 CHARACTERIZATION OF STIFF GRAPHS

The question that we would like to address in this section is: What makes a graph stiff? We wish to derive necessary and sufficient conditions for stiffness in terms of well known graph properties. The most natural of classical graph properties relating to stiffness is connectivity.

3.1.1 Some simple characterizations

We have been able to derive a list of useful properties of stiff graphs:

1. A stiff graph is a block (i.e., has no cut vertex).
2. A cut-set of a stiff graph, separating it into two nontrivial components, must contain at least 3 edges.
3. A 3-cut-set of a stiff graph, separating it into two nontrivial components, must also separate it into two stiff graphs.
4. If $G=\langle V,E\rangle$ is a stiff graph and $V' \subset V$ a vertex cut-set separating G into components $G_i = \langle Vi,Ei\rangle$ and if the subgraph of G

spanned by V' is stiff, then so are the subgraphs spanned by V' ∪ Vi.

5. If v is a vertex of degree two in a graph G, then G is stiff iff the subgraph of G formed by removing v is stiff too.

The above results are a sample from a larger class of results, all of which serve to establish tools for a "divide and conquer" approach to the stiffness problem. For instance, we would like to determine whether a given graph can be torn into stiff subparts which may later be used to synthesize the original graph. Figure 10 depicts a typical configuration, corresponding to result (3) in the above list. Figure 11 depicts another possible configuration which admits tearing, this time a particular instance of (4).

3.1.2 Stiffness and connectivity

Loosely speaking, stiffness is a property of graphs which has to do with the density of edges, i.e., it is a measure of how well different nodes are attached to each other. It is only natural to expect that such a property should bear relation to classical measures of edge-density.

There are three major classical measures of edge-density: node degrees, minimal edge cut-set, minimal vertex cut-set. Here we explore the relations among these three properties and stiffness.

1. <u>There exist graphs whose nodes possess arbitrarily large degrees but which are not stiff.</u>

To prove this result we describe a simple method to construct counter examples. Start with a flexible graph, say a four-bar mechanism. Add nodes and edges to increase the degrees, preserving the flexibility. This process is demonstrated in Figure 12.

The above process serves to show that:

2. <u>There exist graphs with an arbitrarily large minimal edge-cut-set but which are not stiff.</u>

Thus, two of the classical measures of connectivity are not related to stiffness. Let us now consider the strongest measure of connectivity, vertex-connectivity.

We have seen in the previous section that a stiff graph is at least 2-connected. It is easy to construct 2 and 3-connected graphs which are not stiff. But with k-connected graphs, $k \geq 4$, the problem is no longer simple. Indeed any 4-connected graph must have a minimal degree which is at least 4. Thus the total number of edges m is at least twice the number of nodes. Therefore the overall number of internal degrees of freedom $f(G)=2n-m-3$ is not greater than -3. Not only is a 4-connected graph over-constrained, but the

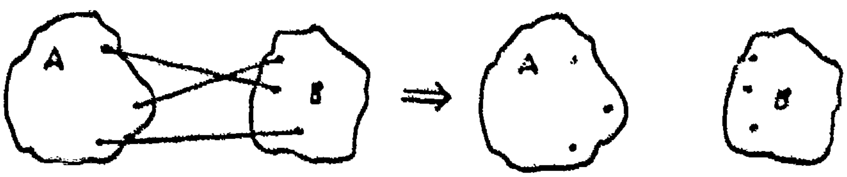

Figure 10. Tearing along a 3-cut-set

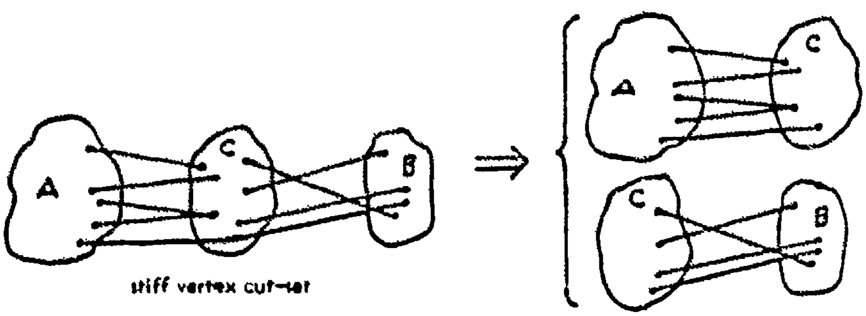

Figure 11. Tearing along a stiff vertex cut-set

connectivity implies that the edges must be well distributed. Intuitively one would expect that a 4-connected graph is stiff.

Not so. It is possible to construct 4-connected and even 5-connected graphs which are flexible. Two such examples are depicted in Figures 13 and 14.

The process we applied to derive these two surprising graphs cannot be applied to produce 6- (or more) connected graphs which are flexible. We do not know at all whether such graphs even exist. The problem is open:

3. <u>Is there a number k such that any k-connected graph is stiff?</u>

At this point, however, the problem is mainly of an academic interest, for a condition which requires such a high connectivity seems to be of no practical significance.

To summarize, we have seen that the relation between stiffness and measures of connectivity (if there is any) are not simple, contrary to the apriori intuition which leads us to explore these relations.

3.2 CONSTRUCTION PROBLEMS

A position-locating algorithm is essentially a process that starts with some set of points whose relative positions are known and gradually attaches new sets of points whose relative positions are computed with respect to the original nucleus. Such a process may be viewed as a (possibly parallel) solidification of parts of the measurement graph into bodies.

To be able to describe incremental construction processes we need to introduce a suitable formalism. In the following we shall describe such a formalism, then apply it to develop and implement construction algorithms.

3.2.1 Stiff hypergraphs

A <u>Hypergraph</u> $H \triangleq \langle V,E\rangle$ consists of a set of vertices V and a set of edges E. An edge is a subset of vertices (not necessarily just two, as in graphs). We shall use this generalized notion of an edge to describe a set of vertices whose relative positions are known. An edge is said to be <u>incident</u> upon a given vertex if it contains the vertex. A vertex is said to be incident upon a

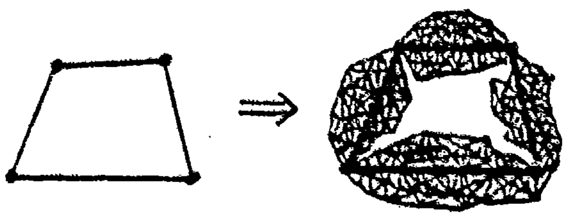

Figure 12. Constructing flexible graphs with arbitrary node degrees

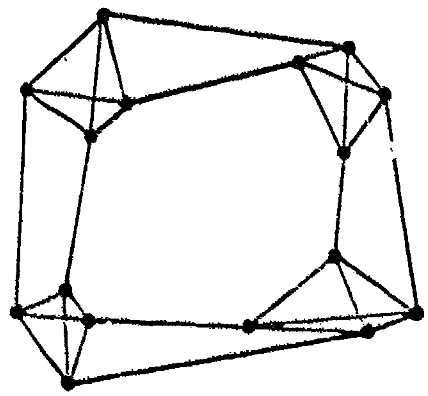

Figure 13. A 4-connected flexible graph

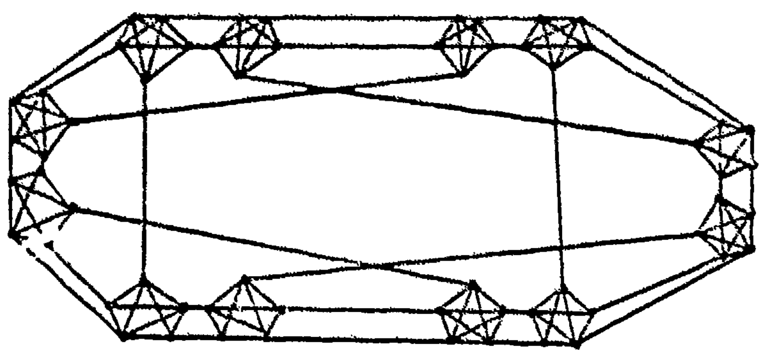

Figure 14. A 5-connected flexible graph

given edge if it is contained in this edge. We shall consider only hypergraphs for which the intersection of any two edges contains at most one point.

To develop a visual intuition of the processes to be discussed, one should consider hypergraphs as a generalization of structures. Rather than considering pin-jointed bars, we consider pin-jointed metal sheets of an arbitrary shape and number of pins (each metal sheet corresponding to an edge of the hypergraph and each vertex corresponding to a joint). To draw further on the analogy, we shall use the term link as an alternative to an edge; the term joint will be employed as an alternative to vertex.

Let d_i denote the degree of the i-th vertex (i.e., the number of edges incident upon it). Let d_i^* denote the degree of the i-th edge (i.e., the number of vertices incident upon it). The degree of the hypergraph H is defined to be the number $d(H) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{m} d_i^*$, where n(m) is the total number of edges (vertices).

The number $f(H) \triangleq 2n+3m-2d(H)$ is called the internal freedom of H. It is easy to verify that for a graph H the definition of f(H) above degenerates to the number of internal degrees of freedom defined in previous sections.

The notion of stiffness can be generalized to hypergraphs as follows. A hypergraph H is said to be stiff iff it contains a spanning hypergraph H' such that

1. $f(H') = 0$

2. for any subhypergraph H" of H' $f(H") \geq 0$

Let us introduce a partial order over hypergraphs, which we call welding. A hypergraph H' is said to be a welding of a hypergraph if each edge of H' is a union of edges in H which span a stiff subhypergraph of H.

It is possible to show that stiffness is preserved under welding. Moreover, each hypergraph possesses a unique welding which is maximal (cannot be welded any more). If M(H) designates the maximal welding of the hypergraph H, then f(M(H)) defines the degrees of freedom of H. It can be shown that $f(M(H)) \geq 3$ with equality iff H is stiff, in which case M(H) has a single edge covering all the nodes.

3.2.2 Incremental construction algorithms

An incremental construction algorithm is a process that starts with a given positioning problem and develops a solution by gradually increasing the sets of points whose relative locations are known. At each stage, the state of the computation may be described as a hypergraph whose edges consist of sets of points whose relative positions are already known. Such a hypergraph is necessarily a welding of hypergraphs in the previous stages. In short, an incremental construction algorithm is a process that traces a chain in the partial order of welding.

We define a welder to be an operator that takes a hypergraph and produces a welding of it. An incremental construction algorithm is thus a process of successive applications of welders.

We have developed software to represent and manipulate structures and hypergraphs. Two types of welders have been implemented and some simple construction algorithms tried. We possess the tools which are necessary to develop position-locating algorithms of increasing sophistication.

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